

## SUMMARY OF DIFFERENTIAL EQUATIONS FOR GROUNDWATER FLOW

All equations are derived by combining Darcy's Law with an equation of continuity (mass balance.)

Symbols used:

b	aquifer thickness
h	piezometric head
K	hydraulic conductivity
S	storage coefficient
$S_s$	specific storage
$S_{ya}$	apparent specific yield
T	transmissibility
t	time
x	} orthogonal coordinate directions
y	
z	

$$\nabla^2 \text{ Laplacian operator } = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Relations:

$$S = bS_s$$

$$T = bK$$

### 1. CONFINED FLOW

#### a. INHOMOGENEOUS ANISOTROPIC CONFINED AQUIFER

$$\frac{\partial}{\partial x} \left( K_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( K_y \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left( K_z \frac{\partial h}{\partial z} \right) = S_s \frac{\partial h}{\partial t} \quad (1)$$

Coordinate system must be selected so that axes are collinear with principal axes of hydraulic conductivity.

If the medium is *homogeneous*, equation (1) reduces to:

#### b. HOMOGENEOUS ANISOTROPIC CONFINED AQUIFER

$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2} = S_s \frac{\partial h}{\partial t} \quad \text{or} \quad \mathbf{K} \cdot \nabla^2 h = S_s \frac{\partial h}{\partial t} \quad (2)$$

Again, the coordinate axes must be chosen to coincide with the hydraulic conductivity axes.

#### c. HOMOGENEOUS ISOTROPIC CONFINED AQUIFER (DIFFUSION EQUATION)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{S_s}{K} \frac{\partial h}{\partial t} \quad \text{or} \quad \nabla^2 h = \frac{S_s}{K} \frac{\partial h}{\partial t} \quad (3)$$

For an aquifer of *constant thickness*,  $b$ , this can be written:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{S}{T} \frac{\partial h}{\partial t} \quad \text{or} \quad \nabla^2 h = \frac{S}{T} \frac{\partial h}{\partial t} \quad (4)$$

Note that because the medium is isotropic, the selection of coordinate axes is arbitrary -- they may be placed in any convenient orientation.

## d. HORIZONTAL FLOW IN HOMOGENEOUS ISOTROPIC CONFINED AQUIFER OF CONSTANT THICKNESS

If flow is horizontal,  $\frac{\partial h}{\partial z} = 0$ , and thus

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T} \frac{\partial h}{\partial t} \quad (5)$$

In this equation piezometric head does not vary with elevation. Note that if a flow has significant *vertical* components of velocity, there does not exist a *single* piezometric surface indicative of flow conditions in the confined aquifer as a whole.

## e. STEADY FLOW IN HOMOGENEOUS ISOTROPIC CONFINED AQUIFER

*Steady flow* means that the flow rate, piezometric head, and amount of fluid in storage do not change with time. Thus the right-hand side of equation (3) becomes zero and we get:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0 \quad \text{or} \quad \nabla^2 h = 0 \quad (6)$$

This is **Laplace's equation**, the subject of much study in other fields of science. Many powerful and elegant methods are available for its solution, especially in two dimensions.

## 2. UNCONFINED FLOW (WATER-TABLE AQUIFERS)

## a. INHOMOGENEOUS ANISOTROPIC UNCONFINED AQUIFER

$$\frac{\partial}{\partial x} \left( K_x \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( K_y \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial z} \left( K_z \frac{\partial h}{\partial z} \right) = 0 \quad (7)$$

The right-hand side equals zero *not* because  $\frac{\partial h}{\partial t} = 0$  (it doesn't), but because in unconfined aquifers

$$S_s \approx 0.$$

The flow domain for which solutions of this equation are sought is not constant because the water-table position changes with time.

## b. HOMOGENEOUS ANISOTROPIC UNCONFINED AQUIFER

$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2} = 0 \quad \text{or} \quad \mathbf{K} \cdot \nabla^2 h = 0 \quad (8)$$

Coordinate axes must coincide with conductivity axes.

## c. HOMOGENEOUS ISOTROPIC UNCONFINED AQUIFER

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = 0 \quad \text{or} \quad \nabla^2 h = 0 \quad (9)$$

Note that this is Laplace's equation again.

d. DUPUIT ASSUMPTIONS FOR UNCONFINED FLOW

Solution of the equations for unconfined groundwater flow is complicated by the fact that the aquifer thickness changes as groundwater is withdrawn; i.e., removal of water from the aquifer lowers the water table.

If vertical components of flow are negligible or small, we can use the *Dupuit assumptions* to simplify the solution of the equations. These assumptions are:

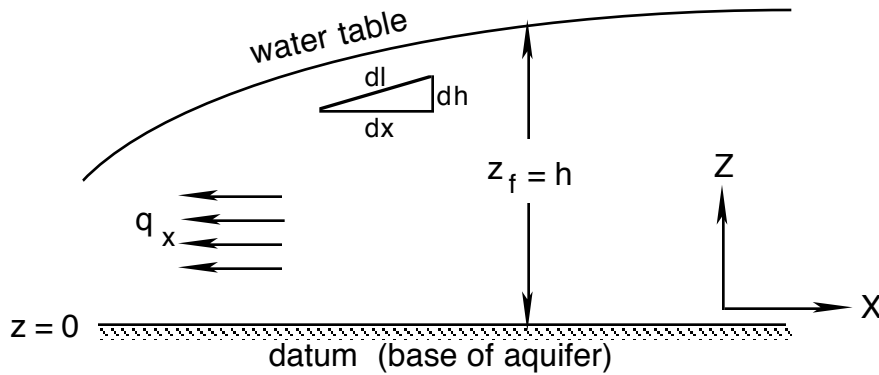
1. assume that the flow is horizontal in any vertical profile
2. assume that the Darcy velocity  $q_x$  is constant over the depth of flow  $z_f$ .  $z_f$  is a function of  $x$ .
3. assume that the Darcy velocity at the free surface (water table) can be expressed as:

$$q = -K \frac{\partial h}{\partial x}, \text{ rather than } q = -K \frac{\partial h}{\partial l}. \text{ This is reasonable for small water-table slopes } \frac{\partial h}{\partial l}.$$

4. these assumptions require a hydrostatic pressure distribution along any vertical line.
5. the consequence of these assumptions is that:

$$Q = -Kh \frac{dh}{dx}.$$

In this relation,  $h$  = piezometric head at water table *and* thickness of flow.



e. BOUSSINESQ EQUATION: HOMOGENEOUS ISOTROPIC UNCONFINED AQUIFER, DUPUIT ASSUMPTIONS

1. Non-linear equation for general case

$$\frac{\partial}{\partial x} \left( h \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( h \frac{\partial h}{\partial y} \right) = \frac{S_{ya}}{K} \frac{\partial h}{\partial t} \quad (10)$$

2. Linearized equation for situations where spatial variation of  $h$  is small with respect to magnitude of  $h$

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S_{ya}}{bK} \frac{\partial h}{\partial t} \quad (11)$$

### 3. AQUIFERS WITH VERTICAL ACCRETION TO FLOW

- a. HOMOGENEOUS ISOTROPIC CONFINED AQUIFER OVERLAIN BY LEAKY CONFINING LAYER (AQUITARD)

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{h_0 - h}{B^2} = \frac{S}{bK} \frac{\partial h}{\partial t} \quad (12)$$

where

$$B = \sqrt{\frac{K_a b_a}{K}} = \text{leakage factor}$$

$K_a$  = hydraulic conductivity of overlying aquitard

$b_a$  = thickness of overlying aquitard

$h_0 - h$  = difference in piezometric head between that of confined aquifer ( $h$ ) and that above aquitard ( $h_0$ )

- b. HOMOGENEOUS ISOTROPIC RECHARGING UNCONFINED AQUIFER

$$\frac{\partial}{\partial x} \left( h \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left( h \frac{\partial h}{\partial y} \right) + \frac{W}{K} = \frac{S_{ya}}{K} \frac{\partial h}{\partial t} \quad (13)$$

where

$W$  = recharge or accretion rate (rate of percolation to water table)